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LETTER TO THE EDITOR

Exact solutions for anharmonic oscillators

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Abstract. We complete an investigation started in an earlier work and present exact solutions and eigenvalues for anharmonic oscillators, the solutions being given in the form of definite integrals. The conditions for the validity of these results are investigated.

In a recent paper (Flessas 1981) we found rigorous solutions for the one-dimensional quantum mechanical doubly anharmonic oscillator defined by the potential

$$V(x)=\omega^2x^2/2+\lambda x^4/4+\eta x^6/6 \qquad \eta>0 \qquad -\infty<x<\infty, \tag{1}$$

in the form of definite integrals. Here we generalise those results and consider anharmonic interactions (Flessas and Das 1980, Magyari 1981)

$$V_N(x)=\frac{\omega^2x^2}{2}+\sum_{n=1}^{2N}a_{n+1}\frac{x^{2n+2}}{2n+2} \qquad a_{2N+1}>0 \qquad (N=1,2,3,\ldots) \tag{2}$$

which may be of interest in various models of the charmonium system. We illustrate the procedure to be followed for

$$V_2(x)=\omega^2x^2/2+a_2x^4/4+a_3x^6/6+a_4x^8/8+a_5x^{10}/10 \qquad a_5>0, \tag{3}$$

in which case the Schrödinger equation reads

$$\left(\frac{\mathrm{d}^2}{\mathrm{d}x^2}+2E-\omega^2x^2-\frac{a_2x^4}{2}-\frac{a_3x^6}{3}-\frac{a_4x^8}{4}-\frac{a_5x^{10}}{5}\right)y(x)=0. \tag{4}$$

The substitutions

$$x^2=t, \qquad y(x)=g(t) \tag{5}$$

transform equation (4) into

$$\left(t\frac{\mathrm{d}^2}{\mathrm{d}t^2}+\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}+\frac{E}{2}-\frac{\omega^2t}{4}-\frac{a_2t^2}{8}-\frac{a_3t^3}{12}-\frac{a_4t^4}{16}-\frac{a_5t^5}{20}\right)g(t)=0. \tag{6}$$

A study of the differential equation (6) reveals that the ansatz

$$g(t)=t^d\exp(at^3+bt^2+ct)\int_{t'\geq 0}^{t}u^e\exp(a'u^3+b'u^2+c'u)\,\mathrm{d}u \tag{7}$$

will provide an exact solution to equation (6). Indeed we obtain, after performing a partial integration, that

$$g(t) = -2\exp(-at^3 - bt^2 - ct) - 4t^{1/2}\exp(at^3 + bt^2 + ct) \times \int_0^t (3au^{3/2} + 2bu^{1/2} + cu^{-1/2}) \exp(-2au^3 - 2bu^2 - 2cu)\,\mathrm{d}u \tag{8}$$

is an exact solution to equation (6) with

$$a = (a_5/5)^{1/2}/6 \qquad (a_5 > 0) \tag{9}$$

$$b = a_4(5/a_5)^{1/2}/32 \tag{10}$$

$$c = (a_3/12 - 5a_4^2/(256a_5))(5/a_5)^{1/2}, \tag{11}$$

provided the conditions

$$a_2 = 6(a_5/5)^{1/2} + 5a_4(a_3/12 - 5a_4^2/(256a_5))/a_5 \tag{12}$$

$$\omega^2 = 20(a_3/12 - 5a_4^2/(256a_5))^2/a_5 + 3a_4(5/a_5)^{1/2}/8 \tag{13}$$

$$E = -3(5/a_5)^{1/2}(a_3/12 - 5a_4^2/(256a_5)) \tag{14}$$

are satisfied. Now since (Gradshteyn and Ryzhik 1965)

$$\int_0^\infty u^{m-1}\exp(-\beta u^2 - \gamma u)\,\mathrm{d}u = (2\beta)^{-m/2}\Gamma(m)\exp\left(\frac{\gamma^2}{8\beta}\right)D_{-m}\left(\frac{\gamma}{(2\beta)^{1/2}}\right), \tag{15}$$

m and β being both positive, and as $a > 0$ (cf equation (9)) the integral in equation (8), denoted henceforth by $I(t)$, exists for $0 \leqslant t \leqslant \infty$; $D_{-m}(w)$ is the parabolic cylinder function.

However, in order that $g(t)$ be physically acceptable we must show that $\lim_{t\to\infty} g(t) = 0$. To facilitate the discussion and to avoid complex quantities in what follows we are going to make the reasonable assumption

$$a_4 > 0. \tag{16}$$

Then by applying the mean value theorem and equation (15) to $I(\infty)$ we have:

$$I(\infty) = \tfrac{1}{4}\pi^{1/2}\exp(c^2/4b - 2at_1^3)(4b)^{1/4}[\tfrac{3}{2}(a_5/5)^2(8/a_4)^{3/2}D_{-5/2}(-z) + D_{-3/2}(-z) - 2zD_{-1/2}(-z)],$$

$$z = -c/(b)^{1/2}, \tag{17}$$

where t_1 originates from the mean value theorem and since $0 \leqslant I(\infty) < \infty$ then $0 \leqslant t_1 = t_1(a, b, c)$. Taking into account now equation (8) and equation (17) we observe that to ensure $\lim_{t\to\infty} g(t) = 0$ it is necessary and sufficient to require

$$z = \frac{V(1, z)}{V(0, z)} + (a_5/5)^2(8/a_4)^{3/2}\frac{V(2, z)}{V(0, z)}, \tag{18}$$

where we used the relation (Abramowitz and Stegun 1970)

$$V(\alpha, z) = \pi^{-1}\Gamma(\tfrac{1}{2} + \alpha)(\sin \pi\alpha D_{-\alpha-1/2}(z) + D_{-\alpha-1/2}(-z)) \tag{19}$$

with $V(\alpha, z)$ being the parabolic cylinder V-function. Equation (18) shows (as $D_{-m} > 0$ for $m > 0$) that $z > 0$ and consequently also

$$c < 0. \tag{20}$$

Condition (18), which appropriately extends relation (14) of Flessas (1981), is solvable as long as $z > V(1, z)/V(0, z)$ and this is definitely the case for $z \geqslant 0.8$. For example, let $z = 1$. Then equation (18) becomes (cf the tables of Abramowitz and Stegun)

$$5/a_5 = (8/a_4)^{3/4} 3.7 \qquad (z = 1). \tag{21}$$

Further, recalling $z = -c/(b)^{1/2}$ and using equations (10)–(11) we obtain

$$1 = (-a_3/12 + 5a_4^2/(256a_5))(5/a_5)^{1/4}(a_4)^{-1/2}(32)^{1/2}. \tag{22}$$

On replacing $5/a_5$ in equation (22) by equation (21) we deduce an equation for a_3 which can be immediately solved for any $a_4 > 0$.

We summarise now our results. Equations (5), (8)–(11) constitute an exact and normalisable solution for equation (4) provided the three relations (12)–(13) and (18) hold between ω, a_2, a_3, a_4, a_5. The eigenvalue is given by expression (14). In practice we can fix arbitrary $a_5 > 0$, $a_4 > 0$ then determine from equation (18) a_3 as described above and finally a_2, ω. As $y(x)$ is nodeless, equation (14) corresponds to the ground state. By multiplying the ansatz (7) with a polynomial in t we may obtain excited states by analogy with the three-dimensional case for equation (2) (Flessas and Das 1980). It is worth noticing that conditions (12)–(13) are markedly different from those following from the ansatz (Flessas and Das 1980) for equation (4)

$$g(t) = t^d \exp(a_1 t^3 + b_1 t^2 + c_1 t). \tag{23}$$

This of course implies that equation (7) is not just the second linearly independent from equation (23) solution to the eigenvalue problem (4). We also remark that Khare (1981) has very recently shown that some of the exact solutions of Flessas and Das (1980) are inaccessible to perturbation theory. Hence we may conjecture that the rigorous results of this work can also be used to test the validity of various approximations applied to general multiterm potentials.

In the case of the potential (2) the ansatz (7) is generalised to

$$g(t) = t^d \exp(\alpha_{N+1} t^{N+1} + \alpha_N t^N + \cdots + \alpha_1 t) \int_{t' \geqslant 0}^{t} u^{e'} \exp(\alpha'_{N+1} u^{N+1} + \cdots + \alpha'_1 u)\, du \tag{24}$$

while $(N+1)$ relations of the type (12)–(13) and (18) are fulfilled.

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